Flight Trajectory Optimization Using Genetic Algorithm Combined with Gradient Method

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Abstract

This paper considers the numerical method to solve trajectory optimization problems where efficient flight trajectories are searched to minimize a performance index with specified constraints. While a wide variety of numerical methods have been developed, the direct application of a nonlinear programming method is frequently used in the recent applications, which transforms the original problem into a nonlinear optimization problem. Although these methods can be applied to complicated applications, it is very difficult to find reasonable initial solutions since practical problems have high nonlinearity. This paper tries to apply the Genetic Algorithm (GA) method. Since the GA approach is one of the random search methods, the initial solutions are selected at random. As shown in numerical examples, the GA approach has slow convergence characteristics and the meaningless fluctuation in the solutions. Therefore, the gradient approach is utilized to refine the solutions. As the gradient method, the authors use the BDH method which approximates the state and control variables using linear interpolation and the collocation method is used to satisfy the dynamic equations. The obtained Hessian matrix can be approximated in a Block Diagonal Form which is suitable for efficient computation. The proposed approach is applied to the optimal accent trajectory optimization of a spaceplane. The spaceplane is a future space transportation vehicle and its conceptual design is investigated.

1 Introduction

Trajectory optimization problems have been widely considered in aerospace engineering fields\(^1\). The problem is formulated as the calculus of variations where an objective function such as a flight time or required fuel is minimized while satisfying initial/final conditions and path constrains for a flying vehicle. Although some techniques such as the dynamic programming of Bellman and the maximum principle of Pontryagin have been developed and used, the recent development of computer is promoting new algorithms\(^2\). Some of the new methods transform the original problem into a nonlinear optimization problem by descritizing the time functions such as control or state variables using a set of discrete variables. Sophisticated algorithms with gradient information for nonlinear optimization problems make it easier to deal with complicated equal or unequal constrains. However, the gradient algorithms require the delicate selection of initial guess of the optimal solution. This paper tries to combine non-gradient techniques with the gradient methods to solve the complicated trajectory optimization methods without accurate initial solutions.

As a non-gradient optimization algorithm, a Genetic Algorithms (GA) is employed, which uses search procedures based on the mechanics of natural genetics, which combines a Darwinian survival-of-the-fittest strategy to eliminate unfit characteristics and uses random information exchange\(^3\). While the original GA codes continuous variables into a string of bit code, the continuous variables can be dealt with by an advanced continuous parameter GA\(^4\) which defines a chromosome as an array of floating point numbers. Since the GA approach is categorized as random search algorithm, initial solutions are selected at random. However, the convergence of the search process is usually very slow.

This paper utilizes the continuous parameter GA in order to select the initial solutions for the BDH method\(^5\) which is one of the gradient methods proposed by us. The characteristics of the GA approach to the trajectory optimization are examined for a simple problem with an analytic solution and the limitation of this approach will be revealed. Additionally, as a practical complicated example, an ascent trajectory optimization problem of a spaceplane is computed by the presented method which uses the GA approach combined with the gradient method. The space plane is a future space transportation vehicle which takes off horizontally, reaches to a space station with a single stage, and lands horizontally\(^6\). In order to perform this mission, the vehicle changes several types of engine, e.g., an air breathing engine and a rocket engine according to its flight phase. The efficient accent trajectory design is strongly desired to maximize payload or minimize consumed fuel.
2 Trajectory Optimization Problem

Defining the state variables \( x(t) \) and the control variables \( u(t) \), the dynamic system is defined by the state equations as

\[
\frac{d}{dt}x(t) = f(x, u, t) \quad (1)
\]

The state and control variables are optimized to minimize the following performance index

\[
J[x, u, t] \quad (2)
\]

while satisfying the initial and final conditions defined as

\[
\psi_l(x(t_l)) = 0, \quad \psi_f(x(t_f)) = 0 \quad (3)
\]

where \( t_l \) and \( t_f \) are initial and final time respectively. In many cases, during a flight path some constrains are introduced as

\[
G(x, u) = 0, \quad H(x, u) \leq 0 \quad (4)
\]

3 GA Approach

The trajectory optimization problem stated in the preceding section can be converted to a nonlinear optimization problem. The time interval from \( t_l \) to \( t_f \) is divided into \( N \) elements to transform the control variables \( u(t) \) as a function of time to a set of discrete variables. By denoting the nodal values of time by \( t_i \) \( (i = 1, 2, ..., N + 1) \), control variables are represented as a set of discrete values \( u_i = u(t_i) \). If the initial states \( x(t_i) \) are specified in the initial conditions, and if the final time \( t_f \) is assumed and the discretized control variables \( u_i \) are assumed, the state variables can be calculated by an appropriate integration scheme. In this paper, the control variables between nodal values of \( u_i \) are interpolated using a third order spline function, and the 4th order Runge-Kutta algorithm is used as the integration scheme. This approximation makes it possible to represent the state variables at time node \( t_i \) as a function of discretized control variables \( u_i \) and the final time \( t_f \). Therefore, the independent variables to be optimized are defined as

\[
p^T = [u_1^T, u_2^T, ..., u_{N+1}^T, t_f^T] \quad (5)
\]

It should be noted that unknown initial states variables can be incorporated into the independent variables.

The performance index \( J \), the final conditions at \( t_f \), and path constrains are defined as a function of the independent variables \( p \). The trajectory optimization problem can be transformed into the following nonlinear optimization problem:

minimize \( J(p) \) \quad (6)

subject to \( \psi_f(p) = 0 \) \quad (7)
\( G(p) = 0 \) \quad (8)
\( H(p) \leq 0 \) \quad (9)

The initial \( N_p \) population of the independent variables is prepared for the continuous parameter GA at random. The generation is updated from the following process:

1) Selection of parents: \( m + 2 \) parents are initially selected at random.

2) Generation of children: the gravity point of the first \( m + 1 \) parents is determined as \( x_p \).

The deviation vectors of each parents are defined as

\[
x_p = \frac{1}{m+1} \sum_{i=1}^{m+1} x_i \quad (10)
\]

\[
d_i = x_i - x_p \quad (11)
\]

The normalized orthogonal base vectors to the deviation vector \( d_i \) \( (i = 1, ..., m) \) are prepared as

\[
e_1, ..., e_{n-m} \quad (12)
\]

where \( n \) is the number of the individual variables \( p \).

Using the deviation vectors and the base vectors, A set of children is generated in the following equation;

\[
x_c = x_p + \sum_{i=1}^{m} w_i d_i + \sum_{i=1}^{n-m} v_i e_i \quad (13)
\]

where \( D \) is the distance of the orthogonal component of the vector \( d_{m+2} \) to the \( d_1, ..., d_m \) vectors, and \( w_i \) and \( v_i \) are random variables with normal deviates. Note that the number of children is specified as \( 2 \times N_c \). The above process is called as UNDX-m method(7) which allows the uniform search.

3) Selection: the fitness function of each parent and generated child is calculated. The fitness function can be generated by using the performance index and the constraints as

\[
F = J + \sum_{i=1}^{m} |w_1^1 \max[0, |G_i(p)| - \epsilon]^2 + w_2^1 \max[0, H_i(p)]^2 \quad (14)
\]

where \( w_1 \) and \( w_2 \) are weighting values in the penalty method which represents the constraints, and \( \epsilon \) is a small number.

The finest chromosome is selected. Another chromosome is chosen from the roulette selection which determines a fine chromosome at random according to the specified probability based on the ranking.

4) Determination of the new generation: According to the MGG (minimum generation gap) method(7), the selected two chromosome are exchanged with the two selected at random from the old parents. Hence the new parents are generated. If the finest solution does not converge, return to the step 2).
4 Gradient Approach

While the many gradient approach methods have been presented for the trajectory optimization problem, this paper uses the BDH method that is one of the direct collocation methods. In the direct collocation method, not only the control variables but also the state variables are decritized. The BDH is using linear interpolation for this discretization. In the similar way as the GA approach, the time is divided into N elements from the initial time \( t_i \) to the final time \( t_f \). The state and control variables are denoted by a set of nodal values as,
\[
x_1, u_1, \ldots, x_N+1, u_{N+1}
\]
and the time interval of the element is given
\[
\Delta t_i = t_{i+1} - t_i
\]

The variables \( X_i \) (i = 1, 2, ..., N + 1) composed of the discretized state and control variables and the time interval are defined as
\[
X_i = [x_i^T, u_i^T, \Delta t_i]^T
\]
The independent variables \( X \) are obtained as,
\[
X = [X_1^T, \ldots, X_{N+1}^T]^T
\]

Using this approximation, the trajectory optimization problem is transformed into a nonlinear programming problem defined as,
\[
\text{minimize } J(X)
\]
subject to
\[
G(X) = 0
\]
\[
x_i + f(x_i, u_i) \frac{\Delta t_i}{2} - [x_{i+1} + f(x_i, u_i) \frac{\Delta t_{i+1}}{2} \frac{K_i}{K_{i+1}}] = 0
\]
\[
K_{i+1} \Delta t_i - K_i \Delta t_{i+1} = 0
\]
\[
\psi(T) = 0
\]
\[
H(X) \leq 0
\]
where \( K_i \) is a constant defined as,
\[
K_i = \Delta t_i / (t_f - t_i)
\]
Note that the Eq. (21) indicates the continuity condition of states at the midpoint in time element \( \Delta t_i \).

In order to solve the nonlinear programming problem, a sequential quadratic programming (SQP) method is employed. The SQP tries to solve the following quadratic programming problem in a sequential manner:
\[
\text{minimize } \nabla J(X^k)d^k + \frac{1}{2}d^k Dd^k
\]
subject to
\[
G'(X^k) + \nabla G'(X^k)d^k = 0
\]
\[
H'(X^k) + \nabla H'(X^k)d^k \leq 0
\]
where \( X^k \) is the vector \( X \) at kth iteration step, \( \nabla \) indicates the gradient of a function associated with the variables \( X \), and \( d^k \) is the update vector \( = X^{k+1} - X^k \). Vector \( G' \) is composed from all the equal constraints. Matrix \( B \) in Eq. (26) denotes the approximation of the Hessian matrix \( H \) for the following Lagrange function
\[
L = J(X) + \nabla X^T G'(X) + \mu^T H(X)
\]
where \( \lambda \) and \( \mu \) are called the Lagrange variables vectors. The SQP formulation assumes the initial \( B \) to be a unit matrix and modifies the matrix \( B \) through iteration steps by using gradient information. The Broyden-Fletcher-Goldfarb-Shanno (BFGS) formulation is used to update the matrix \( B \) in the following way:
\[
B^{k+1} = B^k + \frac{q^k(q^k)^T}{(q^k)^T B^k q^k} - \frac{B^k p^k (p^k)^T B^k}{(p^k)^T B^k p^k}
\]
where
\[
p^k = X^{k+1} - X^k
\]
\[
q^k = \nabla^2 L(X^{k+1}, X^{k+1}, \mu^{k+1}) - \nabla^2 L(X^k, X^k, \mu^k)
\]
The Hessian matrix for Eqs. (19)-(24) is derived as
\[
H = \begin{bmatrix}
\nabla^2 J \quad 0 \\
0 & \\nabla^2 L
\end{bmatrix}
\]
Since the Hessian matrix has a diagonal form, each block in the matrix \( B \) is updated by BFGS formulation independently. The fundamental characteristics of the BDH methods were examined in Reference (10).

5 Numerical Examples

5.1 Rocket Thrust Direction Control

To check the validity of the GA approach, a planar powered flight was investigated. A point mass \( m \) is guided by controlling the direction angle \( \beta \) of a constant thrust with magnitude \( ma \) on a planar inertial coordinate system as shown in Fig.1.

The state equations are
\[
\frac{d}{dt} x = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
0 & a \cos \beta \\
0 & a \sin \beta
\end{bmatrix}
\]
where \( x = [x, y, u, v]^T \)

Let’s consider the transfer of the particle to a specified height \( h \) with the zero vertical velocity in a fixed
given time $T$ so as to maximize the final horizontal velocity. The initial and final conditions and performance index for this problem are

$$\psi_f = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(T) - \begin{bmatrix} h \\ 0 \end{bmatrix} = 0 \quad (37)$$

$$J = -v(T) \quad (38)$$

Dividing the time scale into $N = 20$ elements, the independent variables to be optimized are represented as a set of $\beta_i$.

In this calculation, the constant values are $a = 2$, $h = 20$, and $T = 10$, respectively. The following values are selected in the GA method:

$$N_p = 100, \quad m = 4, \quad N_c = 100 \quad (39)$$

Figure 2 indicates the transition of the finest performance index during the GA process. Figures 3, 4 and 5 are the horizontal velocity, the vertical velocity, and the trajectory, respectively. Those values are compared with analytical solutions. The numerical solutions show fluctuation around the analytical ones. This is remarkable in Fig. 6 showing the control variables, i.e., the time history of the thrust direction. These results indicate that the GA approach has a great advantage in the selection of initial solutions since those are selected at random but has a disadvantage in the convergence characteristics. It should be noted that the fluctuation in the GA method can be reduced if the number of initial population increases and the huge number of generations is allowed.

Figures 7 and 8 show the optimal solutions obtained from the BDH approach in which the initial solutions are obtained from the GA approach. These results indicate the good accuracy of the BDH method.

### 5.2 Space Plane Accent Trajectory Optimization

As a practical and complicated trajectory optimization problem, an accent trajectory of a spaceplane (Fig. 9) is investigated. The space plane is planned as a future space transportation vehicle which takes off horizontally, accents to the space station directly, and descends to the earth to land horizontally. In order to perform this mission in a single stage, the plane is designed to equip with several types of engines to be selected according to the flight environment. A typical accent flight path is shown in Fig. 10 where the space plane rises above 90 km by using an air-turboramjet engine ATR, a scramjet engine SCR, and a rocket engine RE sequentially. The RE is cut-off and the vehicle zooms up to 400 km in the elliptic orbit. At the apogee in the elliptic orbit, the vehicle is put on a circular orbit as shown in Fig. 11.

As shown in Fig. 12, the state variables are the altitude $h = r - R_0$, velocity $v$, flight-path angle $\gamma$ and weight $m$. The control variable is defined as the angle of attack $\alpha$. Note that the engine holds the max thrust. The state equations are defined as

$$\frac{dh}{dt} = v \sin \gamma \quad (40)$$

$$\frac{dv}{dt} = T \cos \alpha - D \quad (41)$$

$$\frac{d\gamma}{dt} = \sin \gamma \left( \frac{v}{m} - \frac{\mu}{r^2} + \frac{\omega^2}{v} \right) \cos \gamma$$

$$+2\omega \quad (42)$$

$$\frac{dm}{dt} = -\frac{T}{I_{sp} g_0} \quad (43)$$

where $\omega$ and $g_0 (= \mu/R_0^2)$ are the angular velocity of the earth and the gravity constant at the earth surface. Additionally, $D$ and $L$ are the lift and the drag. $T$ and $I_{sp}$ are the engine thrust and the specific impulse which are determined according to the Mach number $M$ as

$$M \leq 6 : \quad \text{ATR} \quad (44)$$

$$6 \leq M \leq M_s : \quad \text{SCR} \quad (45)$$

$$M_s \leq M : \quad \text{RE} \quad (46)$$

The aerodynamic characteristics of the spaceplane are determined from Reference (6) and the ATR and SCR engine characteristics are obtained from Reference (10). The other data used are listed in the following table.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Take-off weight</td>
<td>$m_0 = 300$ [ton]</td>
</tr>
<tr>
<td>Wing area</td>
<td>$S_w = 336.8$ [m²]</td>
</tr>
<tr>
<td>ATR intake area</td>
<td>$17.96$ [m²]</td>
</tr>
<tr>
<td>SCR intake area</td>
<td>$17.96$ [m²]</td>
</tr>
<tr>
<td>RE thrust</td>
<td>$T_{RE} = 233.6$ [ton]</td>
</tr>
<tr>
<td>RE $I_{sp}$</td>
<td>$450$ [sec]</td>
</tr>
</tbody>
</table>

Initial conditions are specified to consider take-off conditions at time $t_1 = 0$ as

$$h(0) = 0 \text{ [km]}, \quad v(0) = 150 \text{ [m/s]} \quad (47)$$

$$\gamma(0) = 0 \text{ [deg]}, \quad m(0) = 300 \text{ [ton]} \quad (48)$$

$$\frac{d\gamma}{dt}(0) \geq 0 \quad (49)$$

The Mach number $M_s$ which indicates the switching timing from SCR engine to RE engine is specified as

$$M_s \leq 12 \quad (50)$$

The terminal conditions at $t = t_f$ when the RE is cut-off are specified as

$$h(t_f) \geq 90 \text{ [km]} \quad (51)$$

$$\gamma(t_f) \geq 0 \text{ [deg]} \quad (52)$$

$$H_a[h(t_f), v(t_f), \gamma(t_f)] = 400 \text{ [km]} \quad (53)$$

where $H_a$ is the function which defines the apogee altitude on the elliptic orbit. Several path constraints must
be considered in the following items:

\[
\text{altitude : } h \geq 0 \text{ [km]} \quad (54)
\]
\[
\text{dynamic pressure : } q = \frac{1}{2} \rho v^2 \leq 100 \text{ [kPa]} \quad (55)
\]
\[
\text{angle of attack : } \alpha \leq 20 \text{ [deg]} \quad (56)
\]
\[
\text{heating rate : } Q = 4.4575 \times 10^{-8} \sqrt{\rho v}^{3.07} \leq 300 \text{ [kW/m²]} \quad (57)
\]
\[
\text{load factor : } \frac{L \cos \alpha + D \sin \alpha}{mg} \leq 4 \quad (58)
\]

The objective function to be minimized is the final mass at the circular orbit and is given as:

\[
J = -m(t_f) \exp(- \frac{\Delta v[h(t_f), v(t_f), \gamma(t_f)]}{g_0 I_{sp}}) \quad (59)
\]

where \(\Delta v\) is the velocity change required to put the vehicle on the circular orbit.

As the same way in the previous example, the GA approach is applied to find initial solutions for the BDH method. In the GA calculation, the following data are used:

\[
N_0 = 500, \hspace{0.5cm} m = 10, \hspace{0.5cm} N_c = 100 \quad (60)
\]

Figures 13-16 show the comparison between the GA results after 10,000th generation and the BDH results in which the time scale is divided into \(N = 80\) elements. Note that the initial solutions in the BDH are obtained from the GA method. Figure 13, 14, and 15 are the time histories of the velocity, the altitude and the control variables, respectively. It should be noted that the BDH can refine the solutions. The final mass increases from 75.93 ton to 80.34 ton and the terminal time \(t_f\) is reduced from 1387.8 sec to 835.9 sec. Figure 16 shows the v-h diagram which shows the relation the velocity and the altitude. This clearly indicates the refinement of the trajectory obtained by the BDH which strictly satisfies the constraints of the dynamic pressure and the heating rate.

6 Conclusion

The numerical method for the flight trajectory optimization problem is investigated in this paper. The new approach is combined the Genetic Algorithm (GA) method and the BDH method which is one of the gradient approach. The GA approach is used to find initial solutions for the BDH method. While the GA approach can select initial solution at random, it is difficult to obtain the smooth converged solution. On the other hand, the BDH method has the good convergence characteristics if the good initial solutions are specified. The proposing approach utilizes the good characteristic in each method. The simple example which has an analytic solution is used to check the validity of the method, and the complicated example of the accent flight trajectory optimization of a spaceplane is demonstrated to show the applicability of the proposing approach.

Reference


Figure 1: Coordinate system

Figure 2: History of fitness function vs generation

Figure 3: Time history of horizontal velocity

Figure 4: Time History of vertical velocity

Figure 5: Flight trajectory

Figure 6: Time history of control

Figure 7: Flight trajectory
Figure 8: Time history of control

Figure 9: Spaceplane

Figure 10: Spaceplane accent trajectory

Figure 11: Orbit transfer of spaceplane

Figure 12: State and control variables

Figure 13: Time history of spaceplane altitude
Figure 14: Time history of spaceplane velocity

Figure 15: Time history of spaceplane control

Figure 16: H-V diagram of spaceplane accent trajectory